THE UNIVERSITY OF ALBERTA

WEAKLY CONNECTED FUNCTIONS

by

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ABSTRACT

In Chapter 1 relationships between various classes of functions, more general than Darboux or connected, are discussed. For example $C^{-3} \subset C^{-4}$, where a function $f: X \longrightarrow Y$ is in C^{-3} if $f(C)$ is connected and is in C^{-4} if $f(\overline{C}) \subset \overline{f(C)}$, for each connected subset C of X.

A function is weakly connected if it takes connected, open sets to connected sets. If X is locally connected and $f: X \longrightarrow Y$ is weakly connected, then f is continuous if and only if $f^{-1}(\text{bdry } G)$ is closed for each open subset G of Y. The converse of the above result is also proven.

If X_1 and X_2 are locally connected and $f: X_1 \times X_2 \longrightarrow Y$ is a function weakly connected in each variable separately, then f is weakly connected. It follows that f is continuous if and only if $f^{-1}(\text{bdry } G)$ is closed for every open subset G of Y.

In Chapter 2 functions with closed graphs and dense graphs are characterized in terms of cluster sets and some sufficient conditions are given for a function to have a closed graph. Also some sufficient conditions are given for cluster sets to be connected.

If X is locally connected and Y is rim-compact, then a weakly connected function $f: X \longrightarrow Y$ with a closed graph is continuous.

A theorem is given stating sufficient conditions for a function to be connected. Another theorem states sufficient conditions for a function to be in C^{-3}.

In Chapter 3 it is shown that linear operators and seminorms are weakly connected. This results in some continuity theorems for these functions.

In Chapter 4 some of the results of the previous chapters are extended to multifunctions.

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INTRODUCTION

A Darboux function is a real valued function of a real variable which satisfies the intermediate value property. The expository article [1] discusses these functions in some detail. In 1875 Darboux showed that these functions are not necessarily continuous by proving that every derivative has the intermediate value property and then giving examples of discontinuous derivatives. In [20] , C. H. Rowe characterized continuity of real valued functions of a real variable as follows.

A function f is continuous if and only if:

(i) If x_1 and x_2 are any two points of the domain, then $f(x)$ takes on each value between $f(x_1)$ and $f(x_2)$ in the interval $[x_1, x_2]$ and,

(ii) For every value of y, the set of points $\{x : f(x) = y\}$ is closed.

Since then several authors [2] [13] have presented generalizations of Rowe's result. Theorem 1-26 generalizes all these continuity theorems and is stated as follows.

If X is a locally connected space and $f : X \longrightarrow Y$ is a weakly connected function, then f is continuous if and only if $f^{-1}(\text{bdry } N)$ is closed for each open subset N of Y.

One of the consequences of this theorem is Corollary 1-38 which gives necessary and sufficient conditions for a real valued function of two real variables to be continuous.

A function $f : R \times R \longrightarrow R$ is continuous if and only if f is

continuous in each variable separately and there exists a dense subset D of R such that $f^{-1}(t)$ is closed for every t in D.

Theorem 1-32 gives the converse to the continuity theorem for weakly connected functions and thus the following characterization of locally connected spaces results.

A topological space is locally connected if and only if every weakly connected function on X with the property that the inverse image of every closed, nowhere dense set is closed, is continuous.

The remainder of Chapter 1 lists and compares several classes of noncontinuous functions which are defined in terms of certain families of connected sets. In particular, Theorem 1-10(a) shows that a function from class C^{-3} is in class C^{-4}; i.e., if $f: X \longrightarrow Y$ is such that f(C) is connected for every connected subset C of X, then $f(\overline{C}) \subset \overline{f(C)}$ for every connected subset C of X.

In Chapter 2 cluster sets and partial cluster sets are defined for functions in terms of nets rather than sequences as is done in some of the literature and this allows theorems to be proved without the restriction of first countability on the spaces. With the aid of these notions some of the results in the literature are extended to more general spaces and functions.

In Chapter 3 it is shown that linear operators and seminorms on topological vector spaces are always weakly connected functions and this immediately gives some continuity

theorems for these functions. Convex functions also satisfy certain connectedness conditions and this results in Theorem 3-23, a continuity characterization.

In Chapter 4 many of the theorems of Chapter 1 and 2 are extended to multifunctions.

Throughout, unless otherwise stated, all topological spaces will be T_1. If a reference is given in a chapter to a result in the same chapter, then the chapter number is omitted; otherwise, it is included.

A function $f: X \longrightarrow Y$ is called nearly continuous (almost continuous) at x if $\overline{f^{-1}(N)}$ is a neighborhood of x for every neighborhood N of f(x). If f is nearly continuous at each x in X, then f is called nearly continuous.

CHAPTER ONE

GENERALIZED CONNECTED FUNCTIONS

Many classes of noncontinuous functions are defined in terms of the properties that the image (or inverse image) of a connected set has, where the connected set considered is from a given subfamily of all the connected sets. For example, the real valued functions on the real line with the intermediate value property are exactly those functions which take closed intervals in the domain to intervals in the range.

The purpose of this chapter is to list, compare and investigate properties of several such classes of functions on general topological spaces. In particular, the class of weakly connected functions, defined below, is studied here and the main results with respect to this class of functions are given in Theorem 26 and Theorem 32. Throughout, any topological space considered is to be a T_1-space, unless stated otherwise, and will be denoted by X or Y. In the following eight definitions f is a function from a topological space X to a topological space Y.

1-1 DEFINITION. If f(K) is connected for each connected subset K of X, then f is called a connected function or is said to belong to the class C^{-2}.

1-2 DEFINITION. Let X be a locally connected space and let

$\mathcal{B}$ be any family of connected subsets of X such that the subfamily $\mathcal{B}(x) = \{B \in \mathcal{B} : B$ is a neighborhood of $x\}$ is a local base at x for each x in X. If $f(\overline{B})$ is connected for each B in $\mathcal{B}$, then f is called <u>connected($\mathcal{B}$)</u> or <u>connected with respect to $\mathcal{B}$</u>.

1-3 DEFINITION. If f(K) is a connected set for each connected, open set K in X, then f is called a <u>weakly connected function</u>.

1-4 DEFINITION. If for each x in X and each open set G about x there is a neighborhood $U \subset G$ of x such that f(U) is connected, then f is called a <u>locally connected function</u>.

1-5 DEFINITION. If $\overline{f(K)}$ is connected for every connected subset K of X, then f is said to belong to the class C^{-3}.

1-6 DEFINITION. If $f(\overline{K}) \subset \overline{f(K)}$ for every connected subset K of X, then f is said to belong to the class C^{-4}.

1-7 DEFINITION. If K is any closed, connected set in Y and each x not in $f^{-1}(K)$ has a neighborhood meeting only finitely many components of $f^{-1}(K)$, then f is said to belong to the class $\mathcal{K}$.

1-8 DEFINITION. If K is any connected set in Y and $f^{-1}(\overline{K}) \supset \overline{f^{-1}(K)}$, then f is said to belong to the class $\mathcal{S}$.

1-9 REMARKS. (a) Connected functions are a generalization of Darboux functions (real valued functions of a real variable with the intermediate value property) and have been studied by many of the authors listed in the references. Only recently have these functions been considered on general topological spaces [16] [19].

(b) Functions connected with respect to a base $\mathcal{B}$ were defined as Darboux($\mathcal{B}$) functions by A.M. Bruckner and J.B. Bruckner in [2] and were studied for the case where the domain is euclidean and the range is a separable metric space. If the domain space is the real line with the usual topology and $\mathcal{B}$ is the base of all open intervals, then a real valued, connected($\mathcal{B}$) function is a Darboux function. Also, a function may be connected with respect to one base $\mathcal{B}$ but not with respect to another equivalent base $\mathcal{B}'$, as Example 15 below shows. In their study of approximately continuous transformations Goffman and Waterman [8] showed that every such function on a euclidean space X into a separable metric space Y is connected($\mathcal{B}$), where $\mathcal{B}$ is a base of connected sets with certain properties.

(c) For a real valued function f on the real line the condition that f take open intervals to connected sets is not equivalent to the condition that f be connected. For this reason weakly connected functions may not have been studied. Some important classes of weakly connected functions which are not in general connected functions are

linear operators and seminorms and these are studied in a later chapter.

It is not known if there is an interesting class of weakly connected functions which are not connected($\mathcal{B}$) for any base $\mathcal{B}$ in the domain space as in Definition 2. However, any class of connected($\mathcal{B}$) functions on a locally connected, regular space is shown below to belong to the class of weakly connected functions. It is desirable, therefore, to discover properties of weakly connected functions since these will apply to each class of connected($\mathcal{B}$) functions for any $\mathcal{B}$.

(d) The class C^{-3} is defined by Bruckner, Ceder and Weiss [3] for real valued functions of a real variable and is shown to contain the uniform limits of sequences of Darboux functions. In particular, a convex function is a uniform limit of a sequence of Darboux functions. In [3], the class C^{-3} is denoted by U_o.

(e) In D.E. Sanderson [21] it is shown that a function f in class C^{-4} is characterized by the property that components of $f^{-1}(F)$ are closed in X if F is closed. The class of peripherally continuous functions belongs to class C^{-4} and not to class C^{-3} in general ([15] and Example 1.19). A function f of a space X into a space Y is called <u>peripherally continuous</u> if and only if for each point x in X and each pair of open sets U and V containing x and f(x), respectively, there is an open set D in U containing x such that $f(\text{bdry } D) \subset V$.

(f) Fan and Struble [5] define a function f to be <u>connectedness preserving</u> if it is connected and in class $\mathcal{K}$.

(g) It is shown by Sanderson [21] that a function f in class $\mathcal{S}$ is characterized by the property that $f^{-1}(K)$ is closed if K is closed and connected.

1-10 THEOREM. (a) <u>Every connected function belongs to the class C^{-3} and C^{-3} is a subclass of C^{-4}</u>.

(b) <u>Every connected function is weakly connected and every locally connected function is also weakly connected</u>.

(c) <u>Every function in $\mathcal{S}$ is also in $\mathcal{K}$</u>.

Proof: (a) The first statement follows immediately from the fact that the closure of a connected set is connected. Now, suppose that f is in C^{-3}. To see that f is in C^{-4} it is sufficient, by Remark 9(e), to prove that $f^{-1}(F)$ has closed components for any closed set F. Let K be a component in $f^{-1}(F)$. Since K is relatively closed in $f^{-1}(F)$, if x is in $\overline{K}$ but not in K, then x is not in $f^{-1}(F)$. The set $K \cup \{x\}$ is a connected subset of X and since f is in C^{-3}, $\overline{f(K \cup \{x\})}$ is connected. But this set is just $\overline{f(K)} \cup f(x)$ and cannot be connected in a T_1-space since "$f(K) \subset F$" and "f(x) is not in F" imply that $\overline{f(K)}$ and f(x) are separated by two disjoint closed sets F and f(x). Therefore K is closed in all of X.

(b) The first statement is obvious. Now, let f be a locally connected function and suppose that for some connected, open subset K of X, f(K) is separated; i.e., $f(K) = A \cup B$, where $A \neq \emptyset \neq B$ and $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. If

$A_1 = K \cap f^{-1}(A)$ and $B_1 = K \cap f^{-1}(B)$, then $K = A_1 \cup B_1$, $A_1 \neq \emptyset \neq B_1$ and $A_1 \cap B_1 = \emptyset$. Since K is connected we may assume, without loss of generality, that $\overline{A_1} \cap B_1 \neq \emptyset$ and may pick x in $\overline{A_1} \cap B_1$. Since K is open and f is locally connected there is a neighborhood $U \subset K$ of x such that f(U) is connected. Since U intersects both A_1 and B_1, the subset f(U) of f(K) will intersect both A and B. Thus $f(U) \cap A \cap B = \emptyset$ and this contradicts the fact that f(U) is connected. Thus f(K) is connected.

(c) This is proved by D.E. Sanderson [21].

1-11 THEOREM. Let X be a locally connected space.

(a) Every connected function on X is connected($\mathcal{B}$) for every $\mathcal{B}$.

(b) Every weakly connected function on X is a locally connected function.

Proof: (a) This follows from the fact that every member of $\mathcal{B}$ is connected and thus the closure of each member of $\mathcal{B}$ is connected.

(b) Since X is locally connected there exists a base for the topology of X consisting of connected, open sets. The result is now immediate.

1-12 THEOREM. Let X be locally connected and regular.

(a) For any $\mathcal{B}$ a connected($\mathcal{B}$) function on X is a locally connected function, and thus

(b) For any $\mathcal{B}$ a connected($\mathcal{B}$) function on X is a weakly

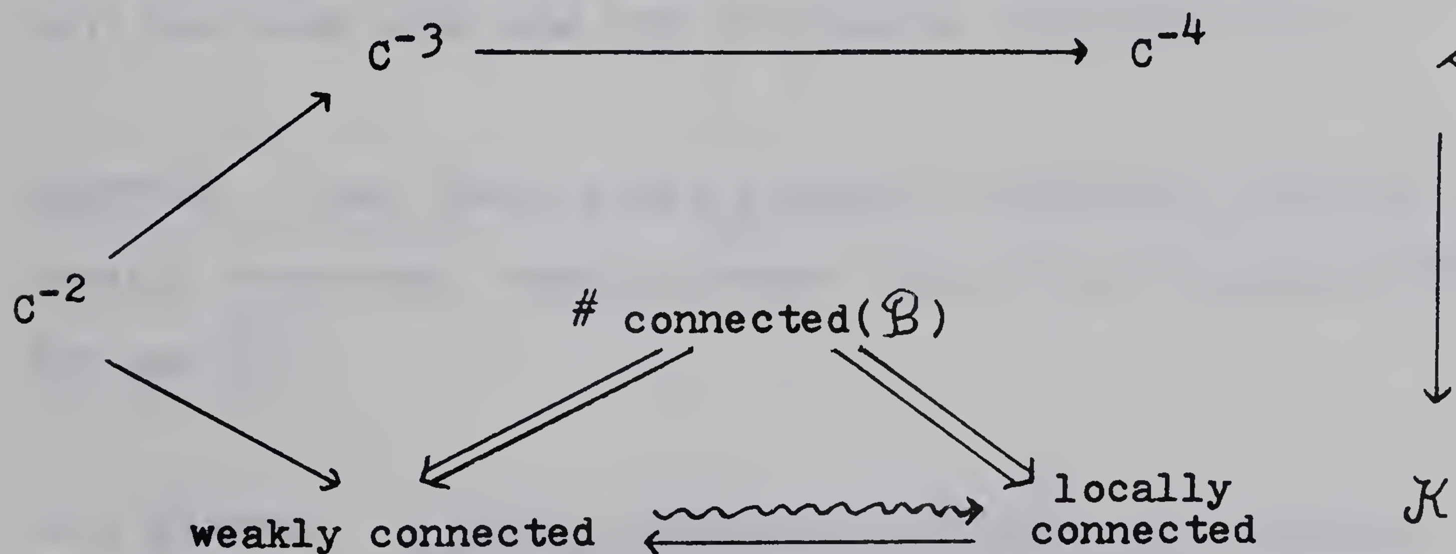
<u>connected function</u>.

Proof: (a) In a regular space the closures of members of $\mathcal{B}(x)$ make a local base for x. Thus for any open set G about x there exists a B in $\mathcal{B}(x)$ such that $\overline{B} \subset G$. Then, since f is connected($\mathcal{B}$), $f(\overline{B})$ is connected.

(b) This follows from (a) and from 10(b).

In the diagram below, which summarizes Theorems 10, 11, and 12, an arrow, say from C^{-3} to C^{-4}, indicates that C^{-3} is a subset of C^{-4}. No arrow indicates no comparison in general and in these cases examples are given below.

1-13 DIAGRAM.



" $\longrightarrow$ " applies to general topological spaces.

" $\rightsquigarrow$ " applies to locally connected spaces.

" $\Longrightarrow$ " applies to locally connected, regular spaces.

This class is defined for locally connected domain only and it is the class of all functions f on X for which there exists a $\mathcal{B}$ such that f is connected($\mathcal{B}$).

In the following examples "R" will denote the real numbers and "I" the closed unit interval $[0,1]$ each with the usual topology.

1-14 EXAMPLE. <u>If the domain space is not regular, then a connected($\mathcal{B}$) function need not be weakly connected or locally connected</u>. Let X be the unit interval with the minimal T_1-topology. This space is locally connected but not regular and if $f: X \longrightarrow I$ is the identity function, then f is connected($\mathcal{B}$) for any $\mathcal{B}$ but is not weakly connected or locally connected. This is so since for any B in $\mathcal{B}$, and for any $\mathcal{B}$, $\overline{B}$ is all of X and since f(X) = I, f is connected($\mathcal{B}$). However, an open set in X has a finite complement and such sets are not in general connected in I.

QUESTION. Does there exist a weakly connected function on a locally connected, regular space which is not connected($\mathcal{B}$) for any $\mathcal{B}$?

1-15 EXAMPLE. <u>A weakly connected function on a locally connected, regular space may be connected with respect to one base $\mathcal{B}$ but not with respect to another base $\mathcal{B}'$. A weakly connected function need not be connected.</u> Let $f: R \longrightarrow R$ be the function defined by $f(x) = \sin 1/x$, if $x > 0$, $f(0) = 1$ and $f(x) = 0$, if $x < 0$. Let $\mathcal{B}$ be the base for the usual topology consisting of all open intervals which do not have the zero as an end point and let $\mathcal{B}'$ be the

equivalent base consisting of all the open intervals. Consider $(-1,0) = B'$ in $\mathcal{B}'$. Since $f(\overline{B'}) = f([-1,0]) = \{0,1\}$, which is not connected, f is not connected($\mathcal{B}'$). Since intervals of the form (a,0) do not occur in $\mathcal{B}$, f is connected($\mathcal{B}$). By Theorem 12(b), f is a weakly connected function. Note that f is not in C^{-4} either.

1-16 EXAMPLE. <u>A function on a locally connected, regular domain may be in C^{-2} and not in $\mathcal{K}$</u>. Define a function $w: I \longrightarrow I$ by

$$w(x) = \limsup \frac{a_1 + a_2 + \cdots + a_n}{n},$$

for $n = 1, 2, 3, \cdots$, where $x = 0.a_1a_2a_3 \cdots$ is the binary expansion of x. The function w is a connected function which takes every interval in I to all of I, (C. Kuratowski, Topologie, Warsaw, 1952). Because of this property the inverse image of any point y is dense in I. Also $w^{-1}(y)$ cannot contain any interval, for then, we would have $w(w^{-1}(y)) = I$. Thus $w^{-1}(y)$ is totally disconnected and any neighborhood of any point not in $w^{-1}(y)$ contains infinitely many components of $w^{-1}(y)$. Consequently, w is not in $\mathcal{K}$.

1-17 EXAMPLE. <u>A function may be in class $\mathcal{S}$ and not be locally connected or in C^{-4}</u>. Any one-to-one function from a T_1-space to a totally disconnected space is in class $\mathcal{S}$, as stated in Sanderson, [21]. In particular, the identity

function from the reals with the usual topology to the reals with the half open (open on the right) interval topology is in $\mathcal{S}$, but is not weakly connected or in C^{-4}.

1-18 EXAMPLE. <u>A function may be in C^{-3} but not be weakly connected or in C^{-2}.</u> The following example is found in F.B. Jones, [11] . Let α, β, γ, $\cdots$ denote a Hamel basis for the real numbers. Every real number x can be expressed uniquely in the form $x = a\alpha + b\beta + c\gamma + \cdots$, where the numbers a, b, c, $\cdots$ are either 0 or rational and at most a finite number of them are different from zero. Define a function f from the reals to the reals by $f(\alpha) = 1$, $f(\beta) = f(\gamma) = \cdots = 0$ and $f(x) = a\, f(\alpha) + b\, f(\beta) + c\, f(\gamma) + \cdots$. Thus $f(x) = a$. Since a, b, c, $\cdots$ are rationals it follows that for each real number x, f(x) is either zero or rational. Also f is of the form $f(x + y) = f(x) + f(y)$ and thus is convex; i.e., $f(\frac{x + y}{2}) \leq \frac{f(x) + f(y)}{2}$. By Theorem 4.5 of Bruckner, Ceder and Weiss, [3] , f is in C^{-3}.

By Theorem 1 of F.B. Jones, [11] , the graph of f is dense in the plane. Thus the image of any open interval (a, b) in R is a dense subset of R consisting of some or all of the rationals. In summary, f((a, b)) is not a single point nor is it a non-degenerate, connected set and so f is not weakly connected.

1-19 EXAMPLE. <u>A function may be in C^{-4} but not in C^{-3}.</u> Define a function $f: R \longrightarrow R$ by $f(x) = 1$, if x is rational and

$f(x) = 0$, if x is irrational. The function f is in C^{-4} since the image of any interval and the closure of the image are always $\{0, 1\}$. From this it also follows that f is not weakly connected or in C^{-3}.

1-20 EXAMPLE. <u>A function may be in $\mathcal{K}$ but not in $\mathcal{S}$.</u> Let x_1 and x_2 be two real numbers such that $x_1 < x_2$ and let $f: R \longrightarrow R$ be a function defined by $f(x_1) = x_2$, $f(x_2) = x_1$ and $f(x) = x$, otherwise. This function is not in $\mathcal{S}$ because if K is the interval $(x_2, x_2 + 1)$, then $f^{-1}(\bar{K}) = \{x_1\} \cup (x_2, x_2 + 1]$ and $\overline{f^{-1}(K)} = [x_2, x_2 + 1]$. It can be readily shown that f is in $\mathcal{K}$.

1-21 EXAMPLE. <u>On non-locally connected spaces a function may be weakly connected, but not locally connected.</u> The identity map from a non-locally connected space onto itself is a homeomorphism and is thus a connected and weakly connected function, but is not a locally connected function.

The remainder of this chapter is concerned with some properties of weakly connected functions. The main results are a continuity theorem for weakly connected functions and a characterization of local connectedness of a space.

1-22 DEFINITION. A function $f: X \longrightarrow Y$ is said to satisfy the <u>property H</u> if each point image $f(x)$ is separable in Y from each closed set C in its complement by a set K (pos-

sibly void) such that $\overline{f^{-1}(K)}$ does not contain x; i.e., $Y-K = A \cup B, f(x) \in A, C \subset B, \overline{A} \cap B = \emptyset = A \cap \overline{B}$ and $A \neq \emptyset \neq B$.

In G.T. Whyburn, [25], the range space Y is defined to be <u>peripherally f-normal</u> provided each point image f(x) is separable in Y from each closed set C in its complement by a set K having a closed inverse under f. <u>Property H of f is implied by peripheral f-normality of Y, but the converse is not true as the following example shows</u>.

1-23 EXAMPLE. With the usual topology on R define a function $f: R \longrightarrow R$ by $f(x) = \sin 1/x$, if $x > 0$, $f(0) = 2$ and $f(x) = 0$, if $x < 0$. R is not peripherally f-normal as can be seen by considering any point $x \in R$ such that $f(x) = 3/4$ and the closed set $C = [0, 1/4] \cup \{2\}$ in the complement of f(x). C and f(x) can be separated only by a set K with 0 in $\overline{f^{-1}(K)} - f^{-1}(K)$ and thus $f^{-1}(K)$ is not closed. To see that f satisfies the property H it is a simple matter to check the two cases $x = 0$ and $x \neq 0$.

Theorem B of G.T. Whyburn, [25] restated for functions is as follows. If X is locally connected, any connected function $f: X \longrightarrow Y$ for which Y is peripherally f-normal, is continuous. The following theorem, then, becomes a generalization. This theorem may be easily extended from functions to multifunctions also, in which case the extension would be a direct generalization of Theorem B.

1-24 THEOREM. <u>If X is locally connected and the weakly</u>

<u>connected function $f: X \longrightarrow Y$ satisfies property H, then f is continuous</u>.

Proof: Let V be any proper open set about an arbitrary point image f(x). Since Y-V is a closed set not containing f(x), there exists, by hypothesis, a set K separating f(x) and Y-V such that $\overline{f^{-1}(K)}$ does not contain x. Let $Y-K = A \cup B$, where f(x) is in A, $Y-V \subset B$ and $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. Since X is locally connected there exists a connected, open neighborhood N of x such that $\overline{f^{-1}(K)} \cap N = \emptyset$. From this it follows that $f(N) \cap K = \emptyset$. Now, f(N) is a connected set and since it contains f(x), which is a subset of V, it cannot intersect Y-V, for otherwise, $A \cup B$ would separate f(N). Thus $f(N) \subset V$ and f is continuous at x. Since x is arbitrary f is continuous.

1-25 COROLLARY. <u>If X is locally connected and f is a weakly connected function on X to a regular space Y, then f is continuous if and only if f^{-1}(bdry G) is closed for each open set $G \subset Y$</u>.

Proof: It is sufficient to show that f has the property H. If C is a closed set not containing f(x), then Y-C is open and there exists an open neighborhood M of f(x) such that $\overline{M} \subset Y-C$. Then K(= bdry M) separates f(x) and C and since $f^{-1}(K)$ is closed, f has the property H. If f is continuous, then f^{-1}(bdry G) is closed since bdry G is closed.

By a proof similar to that of the above theorem, but

not involving the notion of Property H, the above corollary may be strengthened by not requiring regularity of the range space. This result, given below, becomes a generalization of Theorem 3 of Bruckner and Bruckner [2] and part of Theorem C of Klee and Utz [13].

1-26 THEOREM. <u>If X is locally connected and $f: X \longrightarrow Y$ is a weakly connected function, then f is continuous if and only if f^{-1}(bdry G) is closed for each open subset G of Y.</u>

Proof: We show continuity of f at an arbitrary point x. If G is any open set about f(x), then f^{-1}(bdry G) is closed and does not contain x. Since X is locally connected there exists a connected, open neighborhood U of x with $U \cap f^{-1}(\text{bdry } G) = \emptyset$ and by hypothesis f(U) is connected. Now $f(U) \cap (\text{bdry } G) = \emptyset$, but f(U) contains f(x) and since bdry G separates f(x) and Y-G it must be that $f(U) \subset G$.

1-27 REMARKS. The following is also true and the proof is similar to the proof of Theorem 26.

<u>If X is locally connected at x and if $f: X \longrightarrow Y$ is a weakly connected function such that for some local base $\mathcal{M}$ of f(x), f^{-1}(bdry M) is closed in X for each M in $\mathcal{M}$, then f is continuous at x.</u> In fact the result will also hold true if we require only that $\overline{f^{-1}(\text{bdry } M)}$ does not contain x.

1-28 COROLLARY. <u>If X and Y are both locally connected and $f: X \longrightarrow Y$ is a one-to-one, onto function such that both f and</u>

f^{-1} are weakly connected and if both f and f^{-1} take closed, nowhere dense sets to closed sets, then f is a homeomorphism.

Proof: For any open set N the boundary of N is a closed, nowhere dense set and thus Theorem 26 applies to both f and f^{-1}.

1-29 COROLLARY. If f is a real valued, weakly connected function on a locally connected space X, then f is continuous if and only if $f^{-1}(t)$ is closed for each t in R.

In Lipinski [14], it is shown that if X is locally connected and $f: X \longrightarrow R$ is a connected function for which there exists a dense subset D of R such that $f^{-1}(y)$ is closed for each y in D, then f is continuous. As a further corollary to Theorem 26 and the last of the Remarks 27, we have the following extension of Lipinski's result.

1-30 COROLLARY. Let X be locally connected and let $f: X \longrightarrow R$ be a weakly connected function. Then f is continuous if and only if there exists a dense subset D of R such that for each open subset V about f(x), for each x in X, there exists a subset K of D, consisting of at most two points, separating f(x) and R-V such that $\overline{f^{-1}(K)}$ does not contain x.

Proof: The sufficiency of the condition follows by noting that it is possible to choose a local base at f(x) consisting of open intervals with end points in D and then applying the second of Remarks 27.

It is natural now to ask the following question which is a slight variant of that posed by Lipinski in [14].

QUESTION. Is X locally connected if each weakly connected real valued function on X is continuous whenever it has the property as in Corollary 29 or 30?

For the case where X is a metric space a positive answer is given by Klee and Utz [13]. In [23] A.K. Steiner proved further that the answer is in the affirmative if X is completely regular. A partial solution for the above question is also obtainable if it is assumed that each component of X is locally connected to begin with.

To see this let K be a component of X and define a function $f: X \longrightarrow R$ by $f(x) = 1$ if x is in K and $f(x) = 0$ if x in not in K. The conditions of Corollary 30 are satisfied by f and thus f is continuous. Therefore $f^{-1}(0)(= X-K)$ is closed and K is open. Similarly every other component of X is open and thus X is locally connected.

1-31 THEOREM. <u>A space X is locally connected if and only if every weakly connected function on X is locally connected</u>.

Proof: If X is locally connected, then by Theorem 11(b) every weakly connected function is locally connected. If X is not locally connected, then the identity function from X onto X is weakly connected, but is not locally connected.

1-32 THEOREM. <u>Let X be a topological space and suppose that every weakly connected function f on X, with the property that the inverse image of every closed, nowhere dense subset of the range is closed, is a continuous function. Then X is a locally connected space.</u>

Proof: Let G be any open subset of X and let K be a component of G. Define a topological space Y as follows. Choose two points x and y of X such that x is in K and y is in G — K. Let $Y = (X - G) \cup \{x\} \cup \{y\}$ and let the open sets of Y be Y, $\emptyset$, $\{x\}$, $\{y\}$ and $\{x, y\}$. The resulting topological space is one in which $Y - \{x, y\}$ is the only closed, nowhere dense set and all subsets are connected except the set $\{x, y\}$. Define a function $f: X \longrightarrow Y$ by $f(a) = a$, for a in $X - G$, $f(a) = x$, for a in K, and $f(a) = y$, for a in $G - K$. This function is weakly connected(in fact connected) and the inverse image of every closed, nowhere dense subset of Y is closed in X. By hypothesis, then, f is continuous and thus $f^{-1}(x)(= K)$ is open, since $\{x\}$ is open in Y. Since G and K were arbitrary it follows that components of open sets are open in X, making X locally connected.

1-33 COROLLARY. <u>A topological space X is locally connected if and only if every weakly connected function on X, with the property that the inverse image of every closed, nowhere dense set is closed, is continuous.</u>

Proof: If X is locally connected, Theorem 26 applies since

a set is closed and nowhere dense if and only if it is the boundary of an open set. The converse is given in the previous theorem.

Relationships among the various classes of functions defined earlier are desirable and the next chapter will investigate this more fully. The following theorem is of that nature and indicates when a weakly connected real valued function of a real variable is in class C^{-2}.

1-34 THEOREM. <u>A function $f: R \longrightarrow R$ is in class C^{-2} if and only if it is weakly connected and in class C^{-4}.</u>

Proof: Every connected function is weakly connected and in class C^{-4}, by Theorem 10. Conversely, if f is weakly connected and in class C^{-4}, consider an interval of the form $[a, b)$. We have $f((a, b)) \subset f([a, b)) \subset f([a, b]) \subset \overline{f((a, b))}$ and since $f((a, b))$ is connected by hypothesis, $f([a, b))$ is also connected. The argument is the same for any other interval.

Pervin and Levine [19] showed that if $f: X_1 \times X_2 \longrightarrow Y$ is a connected mapping, with all spaces Hausdorff, then f is connected in each variable separately; i.e., $f(x, B)$ is connected for all x in X_1 and all sets B connected in X_2 and $f(A, y)$ is connected for all y in X_2 and all sets A connected in X_1. The converse of this is not true. The following is a similar theorem about weakly connected functions.

1-35 THEOREM. Let X_1 and X_2 be locally connected spaces and let $f: X_1 \times X_2 \longrightarrow Y$ be a function which is weakly connected in each variable separately; i.e., $f(x_1, C_2)$ and $f(C_1, x_2)$ are connected for each connected, open subset C_i of X_i and for each x_i in X_i, $i = 1, 2$. Then f is weakly connected.

Proof: If $U_1 \subset X_1$ and $U_2 \subset X_2$ are connected, open sets, then $U_1 \times U_2$ is connected, open in $X_1 \times X_2$ and $f(U_1 \times U_2)$ is connected. This is so because $f(U_1 \times U_2) = f\left[\left\{\cup(x \times U_2)\right\} \cup (U_1 \times x_o): x \in U_1\right\}\right] = \bigcup\{f(x \times U_2): x \in U_1\} \cup f(U_1 \times x_o)$ for a fixed x_o in U_2. This is a union of connected sets each having a non-empty intersection with a connected set $f(U_1 \times x_o)$. Now $X_1 \times X_2$ is locally connected and each point (x_1, x_2) in $X_1 \times X_2$ has a neighborhood base consisting of sets of the form $U_1 \times U_2$ as above. Thus f is a locally connected function and by Theorem 10(b), f is a weakly connected function.

The converse of the above theorem is not true as the following example shows.

1-36 EXAMPLE. A weakly connected function of two variables which is not weakly connected in each variable separately. Define a function $f: I \times I \longrightarrow I$ by $f(x, y) = y$, if $y \neq 0$ and $f(x, 0) = w(x)$, where w is the function defined in Example 16. This function is weakly connected since if G is a connected, open set in $I \times I$ which does not intersect $\{(x, y): y = 0\}$, then f projects G to a connected set in I. If G intersects $\{(x, y): y = 0\}$, then the intersection

contains an open interval. The image under f of this open interval, and thus of G, is all of I. The function is not weakly connected in each variable separately, since $\{1\} \times [0, 1/2)$ is a connected set and $[0, 1/2)$ is connected, open in I, but the image of this set is not connected since $f((1, 0)) = w(1) = 1$ and $f(\{1\} \times (0, 1/2)) = (0, 1/2)$.

1-37 COROLLARY. Let X_1 and X_2 be locally connected spaces and let $f: X_1 \times X_2 \longrightarrow Y$ be a function which is weakly connected in each variable separately. Then f is continuous if and only if $f^{-1}(\text{bdry } G)$ is closed for every open subset G of Y.

1-38 COROLLARY. A real valued function of two real variables is continuous if and only if it is continuous in each variable separately and there exists a dense subset D of the reals such that $f^{-1}(t)$ is closed for every t in D.

CHAPTER TWO

CLUSTER SETS AND GENERALIZED CONNECTED FUNCTIONS

The notion of the cluster set of a function at a point has been used in analysis; see [1], [3], and has recently been extended to functions on spaces more general than euclidean spaces; see [16] and [19]. In this chapter cluster sets and partial cluster sets are defined for functions in terms of nets rather than sequences as is done in some of the references and this allows theorems to be proved without the restriction of first countability on the spaces. With the aid of these notions some of the results in the literature are extended to more general spaces and functions.

A net will be denoted by (x_d), (or $(x(d))$), where D is the directed domain of the net and x_d is the value of the net at d in D. Usually the directed set is understood, and to conserve on notation it will not be mentioned.

2-1 DEFINITION. The <u>cluster set at x</u> of a function $f: X \longrightarrow Y$ is the set of all y in Y for which there exists a net (x_d), converging to x, such that the net $(f(x_d))$ converges to y. This set is denoted by $C(f;x)$ and we note that $f(x)$ is in $C(f;x)$ for every x in X.

2-2 DEFINITION. Let A be any subset of X and let $f: X \longrightarrow Y$ be a function. The <u>partial cluster set</u> of f at x with

respect to A, denoted by $C_A(f;x)$, is the set of all y in Y for which there is a net (x_d) in A converging to x such that the net $(f(x_d))$ converges to y. Note that a partial cluster set with respect to some set may be empty.

2-3 DEFINITION. For a function $f: X \longrightarrow Y$ and for y in Y, let $T(f;y)$ be the set of all x in X such that there exists a net (x_d) converging to x for which the net $(f(x_d))$ converges to y. Note that if f is not onto, then $T(f;y)$ may be empty for some y not in $f(X)$.

2-4 THEOREM. <u>For a function $f: X \longrightarrow Y$ and for any subset A of X, $C_A(f;x) = \bigcap\{\overline{f(N \cap A)} : N \text{ is in } \mathcal{N}\}$, where $\mathcal{N}$ is a system of neighborhoods of x.</u>

Proof: If z is in $C_A(f;x)$, then there exists a net (x_d) in A which converges to x such that the net $(f(x_d))$ converges to z. Now, (x_d) is in $N \cap A$ eventually for every neighborhood N in $\mathcal{N}$ and so $(f(x_d))$ is in $f(N \cap A)$ eventually for every N. But this means that z is in $\overline{f(N \cap A)}$ for every N, or z is in $\bigcap\{\overline{f(N \cap A)} : N \text{ is in } \mathcal{N}\}$.

Conversely, z in $\bigcap\{\overline{f(N \cap A)} : N \text{ is in } \mathcal{N}\}$ implies that z is in $\overline{f(N \cap A)}$ for every N in $\mathcal{N}$. For every neighborhood M of z we can pick a point $z(M,N)$ in $f(N \cap A) \cap M$ and a point $x(M,N)$ in $N \cap A$ such that $f(x(M,N)) = z(M,N)$. The net $(x(M,N))$ is in A and converges to x, and the net $(z(M,N))$ converges to z. Thus z is in $C_A(f;x)$.

2-5 COROLLARY. <u>If $\mathcal{N}$ is any neighborhood system of a point x in X, then $C(f;x) = \bigcap\{\overline{f(N)} : N \text{ is in } \mathcal{N}\}$. Every partial cluster set and every cluster set of a function at a point is a closed set in X.</u>

2-6 THEOREM. <u>For a function $f: X \longrightarrow Y$ and for any y in Y, $T(f;y) = \bigcap\{\overline{f^{-1}(M)} : M \text{ is in } \mathcal{M}\}$, where $\mathcal{M}$ is a neighborhood system of y in Y.</u>

Proof: If x is in $T(f;y)$, then there exists a net (x_d) converging to x such that $(f(x_d))$ converges to y. Let $\mathcal{M}$ be a neighborhood system of y. Then $(f(x_d))$ is eventually in M for every M in $\mathcal{M}$ and thus (x_d) is eventually in $f^{-1}(M)$ for every M in $\mathcal{M}$. Since (x_d) converges to x, x is in $\overline{f^{-1}(M)}$ for every M. Therefore x is in $\bigcap\{\overline{f^{-1}(M)} : M \text{ is in } \mathcal{M}\}$.

If z is in $\bigcap\{\overline{f^{-1}(M)} : M \text{ is in } \mathcal{M}\}$, then z is in each $\overline{f^{-1}(M)}$, where $\mathcal{M}$ is a neighborhood system of some y in Y. If $\mathcal{N}$ is a neighborhood system of z, then $N \cap f^{-1}(M) \neq \emptyset$ for any N in $\mathcal{N}$ and M in $\mathcal{M}$. For each ordered pair (M,N), we can thus choose a point x(M,N) in N such that f(x(M,N)) is in M. The resulting net (x(M,N)) converges to z and the net (f(x(M,N))) converges to y. Therefore z is in $T(f;y)$.

2-7 COROLLARY. <u>For any function $f: X \longrightarrow Y$ and for any y in Y, the set $T(f;y)$ is closed in X.</u>

2-8 THEOREM. <u>For any subset A of X and for a function $f: X \longrightarrow Y$, $C_A(f;x) \subset \overline{f(A)}$.</u>

Proof: If y is in $C_A(f;x)$, then there is a net (x_d) in A which converges to x and the net $(f(x_d))$ converges to y. Since $f(x_d)$ is in f(A) for each d, it follows that y is in $\overline{f(A)}$.

2-9 THEOREM. <u>The graph of a function $f: X \longrightarrow Y$ is closed in $X \times Y$ with the product topology if and only if $C(f;x) = f(x)$ for each x in X.</u>

Proof: The graph of f is closed if and only if whenever (x_d) is a net in X which converges to x and $(f(x_d))$ converges to y, then $y = f(x)$. The result follows readily from this statement.

2-10 THEOREM. <u>For any function $f: X \longrightarrow Y$ the graph of f is closed in $X \times Y$ if and only if $T(f;y) = f^{-1}(y)$ for all y in Y.</u>

Proof: Suppose that $T(f;y) = f^{-1}(y)$ for every y in Y and let (x_d) be a net which converges to x with the net $(f(x_d))$ converging to y. Then x is in $T(f;y)$ and consequently, $y = f(x)$. This means that the graph of f is closed.

Conversely, suppose that the graph of f is closed. Since it is always true that $f^{-1}(y) \subset T(f;y)$, we need show only that $T(f;y) \subset f^{-1}(y)$, for any y in Y. For any x in $T(f;y)$ there exists a net (x_d) converging to x such that the net $(f(x_d))$ converges to y. Since the graph is closed, we have $y = f(x)$ or x is in $f^{-1}(y)$ and consequently, $T(f;y) \subset f^{-1}(y)$.

2-11 THEOREM. The graph of a function $f: X \longrightarrow Y$ is dense in $X \times Y$ if and only if $C(f;x) = Y$ for each x in X.

Proof: Suppose $C(f;x) = Y$ for each x in X. For any (x,y) in $X \times Y$ let $U_x \times V_y$ be a neighborhood of (x,y), where U_x is a neighborhood of x and V_y is a neighborhood of y. By hypothesis, y is in $C(f;x)$. So there exists a net (x_d) converging to x such that $(f(x_d))$ converges to y. Since (x_d) is eventually in U_x and $(f(x_d))$ is eventually in V_y, it follows that $((x_d, f(x_d)))$ is eventually in $U_x \times V_y$. Therefore the graph of f is dense in $X \times Y$.

Conversely, suppose that the graph of f is dense in $X \times Y$. For arbitrary x in X and y in Y it suffices to show that y is in $C(f;x)$. For every neighborhood about (x,y) there is a member $(x_d, f(x_d))$ of the graph of f in that neighborhood. The net $((x_d, f(x_d)))$, so chosen, converges to (x,y) and so (x_d) converges to x and $(f(x_d))$ converges to y. This means that y is in $C(f;x)$.

Some relationships between cluster sets and partial cluster sets are now investigated. If $\mathcal{A}$ is a certain family of subsets of a space X, then the collection of all $C_A(f;x)$ such that A is in $\mathcal{A}$ and x is in X is called the family of all cluster sets on $\mathcal{A}$. If X itself is in $\mathcal{A}$, then the collection contains all the cluster sets $C(f;x)$. Throughout, it is understood that the function $f: X \longrightarrow Y$ is fixed.

2-12 REMARKS. Let $C(X)$ denote the family of all non-

degenerate, connected subsets of X and suppose that X is such that C(X) is not empty. The conditions of "cluster sets on C(X) being connected" and of "cluster sets being connected" are not related in general. The following is an example where the cluster set of a function f at a point is connected for every point in the domain of f, but for some of the points the cluster sets on C(X) may not all be connected.

Let $f: R \longrightarrow R$ be the function of Example 1-15, defined by $f(x) = \sin 1/x$, if $x > 0$, $f(0) = 1$ and $f(x) = 0$, if $x < 0$. It is easy to see that for each x in R, $C(f;x)$ is connected. However, $[-1, 0]$ is in C(R) and $C_{[-1, 0]}(f;0) = \{0, 1\}$ which is not a connected subset.

The following example shows that all the cluster sets on C(X) may be connected yet there exists an x in X such that $C(f;x)$ is not connected.

Let X be the following subset of the plane with the relative usual topology. $X = \bigcup\{A_n : n = 0, 1, 2, \cdots\}$,

where $A_o = \{(x, y) : -1 \leq x \leq 1, y = 0\}$, and

$A_n = \{(x, y) : 1/n \leq |x| \leq 1, y = 1/n\}$,

for $n = 1, 2, 3, \cdots$.

Let Y be the following subset of the plane with the relative usual topology. $Y = \{(x, y) : |x| = 1, y = 1/n, n = 1, 2, \cdots\} \cup \{(-1, 0), (1, 0)\} \cup \{(0, 0)\}$. Define a function $f: X \longrightarrow Y$ by $f(z) = (0, 0)$, for z in A_o, $f(z) = (-1, 1/n)$, if z is in $\{(x, y) : -1 \leq x \leq -1/n, y = 1/n\}$, and $f(z) = (1, 1/n)$, if z is in $\{(x, y) : 1/n \leq x \leq 1, y = 1/n\}$. This is a weakly

connected function and we note further that $C(X)$ consists of all nondegenerate intervals. It is easy to see that for any z in X and for any A in $C(X)$, $C_A(f;z)$ is either $\emptyset$ or $f(z)$ and is thus connected. However, if $z = (0, 0)$, then $C(f;z) = \{(-1, 0), (0, 0), (1,0)\}$, which is not a connected set.

The following theorem shows that for a large class of spaces the two notions are comparable.

2-13 THEOREM. Let X be a space in which all components are open; in particular, X may be connected or locally connected. If cluster sets on the family of all connected, open sets are all connected, then the cluster set of a function at a point is connected for each point in X.

Proof: Each x in X is contained in some connected, open set K which is taken to be the component containing x. It is clear that $C_K(f;x) \subset C(f;x)$. If y is in $C(f;x)$, there exists a net (x_d) converging to x such that $(f(x_d))$ converges to y and since (x_d) is eventually in K, y is in $C_K(f;x)$. Therefore $C_K(f;x) = C(f;x)$ and since $C_K(f;x)$ is connected, $C(f;x)$ is connected.

2-14 THEOREM. Let f be a function from the reals to the reals and denote by $\mathcal{A}^+$ the set of all non-degenerate intervals in R which contain a certain fixed point x as a left end point. Define $\mathcal{A}^-$ similarly for the same point x. If $C_A(f;x)$ is connected for every A in $\mathcal{A}^+ \cup \mathcal{A}^-$, then $C(f;x)$ is

<u>connected</u>.

Proof: For any A in $\mathcal{A}^+$, $C_A(f;x) \subset C(f;x)$ and for any A' in $\mathcal{A}^-$, $C_{A'}(f;x) \subset C(f;x)$. Therefore, $C_A(f;x) \cup C_{A'}(f;x) \subset C(f;x)$. If y is in C(f;x), there exists a net (x_d) converging to x such that $(f(x_d))$ converges to y. There is a subnet (x_b) of (x_d) such that (x_b) converges to x from the right (without loss of generality). Since $(f(x_b))$ converges to y, it follows that y is in $C_A(f;x)$ and thus we have $C_A(f;x) \cup C_{A'}(f;x) = C(f;x)$. The element f(x) is common to the two connected terms on the left and thus C(f;x) is connected.

The first example in Remark 12 shows that the converse of this theorem is not true.

The following theorem and its corollaries give some sufficient conditions for cluster sets to be connected. The results are extensions of Theorem 3.7 of Pervin and Levine [19] in the sense that the function is more general than the connected function and the topological spaces are more general. Note that a real valued, weakly connected function with a locally connected domain has connected cluster sets.

2-15 DEFINITION. A function $f: X \longrightarrow Y$ is said to be <u>subcontinuous</u> if whenever a net (x_d) converges to some x in X, then the net $(f(x_d))$ has a subnet which converges to some y in Y.

Subcontinuity of a function was defined by R.V. Fuller [6].

2-16 THEOREM. Let X be locally connected and let Y be a normal space. If $f: X \longrightarrow Y$ is a function which is

(i) weakly connected (or in C^{-3}), and

(ii) subcontinuous,

then $C(f;x)$ is connected for every x in X.

Proof: Suppose that for some x, $C(f;x)$ is the disjoint union of two non-empty sets A and B such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. Since $C(f;x)$ is closed both A and B are closed and by normality of Y there exist two disjoint open sets O_A and O_B, containing A and B, respectively, whose closures are also disjoint. If $\mathcal{N}$ is the neighborhood system of x consisting of connected, open sets, then $C(f;x) = \cap \{\overline{f(N)} : N \text{ is in } \mathcal{N}\}$ and each $\overline{f(N)}$ is connected. If f is weakly connected it is immediate that for each N in $\mathcal{N}$, $f(N) \not\subset O_A \cup O_B$. When f is in C^{-3} we use the following argument. If $f(N) \subset O_A \cup O_B$, then $\overline{f(N)} \subset \overline{O_A \cup O_B} = \overline{O_A} \cup \overline{O_B}$. But since $\overline{f(N)}$ is connected, it must be contained in one and only one of $\overline{O_A}$ or $\overline{O_B}$. If $\overline{f(N)} \subset \overline{O_A}$, then $\overline{f(N)} \cap O_B = \emptyset$ and this is impossible since $B \subset \overline{f(N)} \cap O_B$. Similarly, $\overline{f(N)}$ cannot be contained in $\overline{O_B}$. Thus $f(N) \not\subset O_A \cup O_B$.

For each N in $\mathcal{N}$ choose a point $y(N)$ in $f(N) \cap (Y - (O_A \cup O_B))$ and an $x(N)$ in N such that $f(x(N)) = y(N)$. The net $(x(N))$ so obtained converges to x. By hypothesis the net $(f(x(N)))$ has a subnet which converges to some point y in Y and since this subnet is in $Y-(O_A \cup O_B)$, which is a closed set, it follows that y is in $Y - (O_A \cup O_B)$. The corresponding subnet of $(x(N))$ converges to x and thus y is in

$C(f;x)$. This contradicts $C(f;x) \subset O_A \cup O_B$ and thus we conclude that $C(f;x)$ is connected.

The above theorem cannot be extended to the class of functions C^{-4} as Example 1-19 shows.

2-17 COROLLARY. Let X be locally connected and Y a compact, T_2-space. If $f: X \longrightarrow Y$ is a weakly connected function, or is in C^{-3}, then $C(f;x)$ is connected for each x in X.

Proof: This follows readily since a compact, T_2-space is normal and f is subcontinuous since Y is compact.

With a few obvious modifications in the proof of Theorem 2-16 we obtain the following results.

2-18 THEOREM. If X is a locally connected, locally compact space and Y is a T_2-space and if $f: X \longrightarrow Y$ is a weakly connected function (or is in class C^{-3}) which takes compact sets to compact sets, then $C(f;x)$ is connected for each x in X.

2-19 THEOREM. If X is a locally connected, first countable space and Y is a normal, first countable space and if $f: X \longrightarrow Y$ is a weakly connected function, or is in class C^{-3}, and takes compact sets to compact sets, then $C(f;x)$ is connected for each x in X.

Proof: The fact that the set consisting of the elements of

a convergent sequence and a limit point of the sequence is a compact set is used.

The remainder of this chapter is devoted to investigating what combination of conditions on a function in one class make it a function which belongs to another defined class. The classes of functions considered will be those defined in Chapter 1 as well as the classes of continuous functions, connectivity functions and functions with a closed graph. In view of the nature of the domain spaces of most of the examples given in Chapter 1 one would expect that any extra conditions for a function from one class to be in another would have to be those on either the range space or on the function itself.

2-20 THEOREM. <u>Let Y be a locally connected, regular space and let $f: X \longrightarrow Y$ be a function in class $\mathcal{S}$; i.e., $f^{-1}(\bar{K}) \supset \overline{f^{-1}(K)}$ for every connected subset K of Y.</u> <u>Then the graph of f is closed in $X \times Y$.</u>

Proof: By Remarks 1-9(g), $f^{-1}(K)$ is closed for every closed, connected set K. By Theorem 2-10 it is sufficient to show that $T(f;y) = f^{-1}(y)$ for every y in Y. For any y in Y there is a neighborhood system $\mathcal{N}_y$ of y consisting of closed, connected sets N. Then $T(f;y) = \bigcap\{\overline{f^{-1}(N)} : N \text{ is in } \mathcal{N}_y\} = \bigcap\{f^{-1}(N) : N \text{ is in } \mathcal{N}_y\} = f^{-1}(\bigcap\{N : N \text{ is in } \mathcal{N}_y\}) = f^{-1}(y)$ and consequently the graph of f is closed.

2-21 REMARK. In the above proof it was required only that the inverse images of members of a certain neighborhood system of a point be closed and not that the inverse image of all closed, connected sets be closed.

2-22 EXAMPLE. <u>The converse of the above theorem is not true</u>. Let $f: R \longrightarrow R$ be the function defined by $f(x) = 1/x$, for $x > 0$ and $f(x) = -1$, for $x \leq 0$. The graph of f is closed but $f^{-1}([1,\infty)) = (0, 1]$ which is not closed. Therefore f is not in $\mathcal{S}$.

2-23 REMARKS. In Theorem 9 of [21], D.E. Sanderson shows that if Y is semi-locally connected, then a function $f: X \longrightarrow Y$ is continuous if and only if f is in $\mathcal{S}$. Recall that a space is <u>semi-locally connected at y</u> if and only if y has a basis of neighborhoods whose complements have finitely many components. If the space is semi-locally connected at each of its points, it is called <u>semi-locally connected</u>.

Class $\mathcal{S}$ is a subclass of $\mathcal{K}$ and Example 1-20 shows that Theorem 20 cannot be extended to class $\mathcal{K}$. In [21], Theorem 8, it is shown that a function in class $C^{-4} \cap \mathcal{K}$ is in class $\mathcal{S}$. Thus the following is true.

2-24 COROLLARY. <u>If Y is locally connected, regular and $f: X \longrightarrow Y$ is in $C^{-4} \cap \mathcal{K}$, then the graph of f is closed in $X \times Y$.</u>

2-25 COROLLARY. Let Y be locally connected, regular and let $f: X \longrightarrow Y$ be a function in $\mathcal{K}$ which is either

(i) connected,

(ii) peripherally continuous, or

(iii) a connectivity function.

Then the graph of f is closed.

Proof: Each of these classes of functions belongs to the class C^{-4}. See Remarks 1-9(e) for the case of peripherally continuous functions.

2-26 DEFINITION. A topological space is called rim-compact if it is T_2 and has a base for the open sets each member of which has a compact boundary. It is known that a rim-compact space is regular [4].

It is well known that if Y is compact and $f: X \longrightarrow Y$ is a function with a closed graph, then f is continuous. The following theorem shows that the restriction on Y may be reduced somewhat if f is in addition weakly connected and X is locally connected.

2-27 THEOREM. Let X be a locally connected space and Y a rim-compact space. If $f: X \longrightarrow Y$ is a weakly connected function whose graph is closed, then f is continuous.

Proof: By Remarks 1-27, it is sufficient to show that $f^{-1}(\text{bdry } O)$ is closed for every open set O in Y whose boundary is compact. If x is in $\overline{f^{-1}(\text{bdry } O)} \setminus f^{-1}(\text{bdry } O)$, there

exists a net (x_d) in f^{-1}(bdry 0) converging to x. The net $(f(x_d))$ is in bdry 0 which is compact and so there exists a subnet (x_b) of (x_d) such that $(f(x_b))$ converges to some point y in the boundary of 0. Since the graph of f is closed, y = f(x), or x is in $f^{-1}(y)$ which is a subset of f^{-1}(bdry 0). Therefore, f^{-1}(bdry 0) is closed. If f^{-1}(bdry 0) is void, a proof similar to that of Theorem 1-26 suffices.

2-28 EXAMPLE. <u>If the condition that f is weakly connected is omitted from the above theorem, then the remaining conditions are not sufficient to assure continuity.</u> For example, the function $f: R \longrightarrow R$ defined by $f(x) = 1/x$, for $x \neq 0$ and $f(0) = 0$ has a closed graph, but is not weakly connected. The range space is rim-compact.

2-29 COROLLARY. <u>Let X and Y be locally connected, rim-compact spaces. If $f: X \longrightarrow Y$ is a one-to-one, onto function such that both f and f^{-1} are weakly connected and for connected, open sets 0 in Y, $f^{-1}(\overline{0}) \supset \overline{f^{-1}(0)}$, (or $f(\overline{0}) \supset \overline{f(0)}$, for any connected, open set 0 in X), then f is a homeomorphism.</u>

Proof: By Theorem 20 and Remark 21 the graph of f, and consequently of f^{-1} (the graph of f^{-1}, and consequently of f) is closed. By Theorem 27 both f and f^{-1} are continuous.

The following theorem generalizes Theorem 3.8 of [19].

The proof is analogous to that given in [19], but is presented here for completeness.

2-30 THEOREM. <u>If X is locally connected and Y is a compact, T_2-space, then a weakly connected function $f: X \longrightarrow Y$ is continuous at x in X if and only if $C(f;x)$ is either finite or denumerable for each x in X.</u>

Proof: By Corollary 17, $C(f;x)$ is connected and since it is also closed it is a compact subset of Y. In G.T. Whyburn [26], p.16 it is shown that a compact, connected set is never the union of a countable number (greater than 1) of disjoint closed sets. Thus $C(f;x)$ cannot be the union of a countable number of points unless $C(f;x) = f(x)$. So for each x in X, $C(f;x) = f(x)$ and by Theorem 9 the graph of f is closed. Since the range space is compact, f is continuous.

Conversely, if f is continuous, $C(f;x) = f(x)$ for each x and the condition is satisfied.

The following results give some conditions which ensure that a function is connected. These theorems resulted from an attempt to give some characterization to connected functions for certain topological spaces.

2-31 THEOREM. <u>Suppose that a function $f: X \longrightarrow Y$ satisfies the properties</u>

(i) <u>f is subcontinuous,</u>

(ii) <u>$C(f;x)$ is connected for each x in X, and</u>

(iii) *for each non-degenerate, connected subset C of X and for each x in C, $C(f;x) \subset f(C)$.* *Then f is a connected function.*

Proof: Suppose that for some non-degenerate, connected subset C of X, f(C) is not connected but is the union of two non-empty sets A and B such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. If $A_1 = f^{-1}(A) \cap C$ and $B_1 = f^{-1}(B) \cap C$, then $C = A_1 \cup B_1$, $A_1 \cap B_1 = \emptyset$ and A_1 and B_1 are non-empty sets. Since C is connected we may, without loss of generality, pick a point x in $\overline{A_1} \cap B_1$. Then f(x) is in B and there is a net (x_d) in A_1 which converges to x. The net $(f(x_d))$ is in A and by (i) there is a subnet $(f(x_b))$ of $(f(x_d))$ which converges to some point y in Y; in fact y is in $\overline{A}$. Also, y is in C(f;x) since (x_b) converges to x and $(f(x_b))$ converges to y. By (iii) y is in $f(C) = A \cup B$, and in particular, y is in A since $\overline{A} \cap B = \emptyset$. In summary we have $C(f;x) \subset A \cup B$, $C(f;x) \cap A \neq \emptyset$ and $C(f;x) \cap B \neq \emptyset$. But, since A and B are separated, this contradicts hypothesis (ii) and thus f(C) is connected.

2-32 COROLLARY. *Let Y be a compact space and $f: X \longrightarrow Y$ a function such that C(f;x) is connected for each x in X and $C(f;x) \subset f(C)$ for each non-degenerate, connected subset C of X and for each x in C.* *Then f is a connected function.*

Proof: Since Y is compact, (i) of the theorem is also satisfied.

2-33 COROLLARY. *Let Y be a compact, T_2-space and X a locally*

connected space. If $f: X \longrightarrow Y$ is a weakly connected function (or in class C^{-3}) and is such that $C(f;x) \subset f(C)$ for each non-degenerate, connected subset C of X and each x in C, then f is connected.

Proof: By Corollary 17, $C(f;x)$ is connected for each x in X.

2-34 COROLLARY. Let X be locally connected, locally compact (first countable), and let Y be a T_2 (respectively, normal, first countable, T_2) space. If $f: X \longrightarrow Y$ is a weakly connected function which takes compact sets to compact sets and $C(f;x) \subset f(C)$ for each non-degenerate, connected subset C of X and for each x in C, then f is a connected function.

Proof: By Theorem 18 (respectively, Theorem 19), $C(f;x)$ is connected for each x in X. Since f takes compact sets to compact sets (respectively, since a convergent sequence with its limit point is a compact set), the first condition of the theorem is satisfied.

2-35 REMARKS. (a) The conditions in Theorem 31 do not imply that f is a continuous function or even that f is a connectivity function. To show this, a modification of Example 1-16 suffices. Define a function $g: I \longrightarrow I$, ($I = \{x$ in R: $0 \leq x \leq 1$) by $g(x) = 0$, if $x = w(x)$ and $g(x) = w(x)$, otherwise. The function g still takes on each value in I on each interval, but the graph of g does not meet the diagonal $y = x$ and thus is not connected. However, $C(g;x) = I = g(C)$ for each interval C in I. Note also that the graph of g is not

closed since it is dense in $I \times I$.

(b) If the condition (iii), that $C(f;x) \subset f(C)$ is dropped, then the remaining conditions are not adequate to assure that f is connected. Consider the function of Example 1-15. The range of f is compact; thus (i) is satisfied. For each $x \neq 0$, $C(f;x) = f(x)$ and $C(f;0) = [-1,1]$; thus (ii) is satisfied. However, let C be the interval $[-1, 0]$. Since $f(C) = \{0, 1\}$, clearly, $C(f;0) \not\subset f(C)$.

(c) If the second condition that $C(f;x)$ be connected for each x in X is dropped, then the remaining conditions may not be adequate for f to be connected. Consider the function of Example 1-19. For each x in X, $C(f;x) = \{0, 1\}$ and for each interval C in X, $f(C) = \{0, 1\}$. Thus (iii) is satisfied, but (ii) is not. It is clear that (i) is satisfied. Also, f is not a connected function.

(d) Without (i), the remaining conditions are not adequate to assure that f is connected. Consider the function $f: R \longrightarrow R$ defined by $f(x) = 1/x$, if $x \neq 0$ and $f(0) = 0$. Since the graph of f is closed, for each x in X, $C(f;x) = f(x)$ and conditions (ii) and (iii) are satisfied. If $x_n = 1/n$ $(n = 1, 2, 3, \cdots)$, then (x_n) converges to zero, but $f(x_n) = n$ for each n and no subsequence of $(f(x_n))$ converges. Thus (i) is not satisfied and it is clear that f is not a connected function.

(e) The following result is proved in R.V. Fuller [6]. "Let $f: X \longrightarrow Y$ be a function. A sufficient condition that f be continuous is that f have a closed graph and that f be

subcontinuous. If Y is Hausdorff, the condition is also necessary."

If the graph of a function is closed, then the conditions (ii) and (iii) of Theorem 31 are satisfied since for each x, $C(f;x) = f(x)$. In 35(a), it is seen that the converse is not true. We see then that weakening the condition that f have a closed graph, in Fuller's result, to the conditions (ii) and (iii) does not give continuity but does ensure connectedness of f.

QUESTION. Is there some sort of converse to Theorem 31? If f is connected, what two conditions from (i), (ii) or (iii) imply the third? Theorem 16 may be a result in that direction.

2-36 THEOREM. <u>Let $f: X \longrightarrow Y$ be a function such that</u>

(i) <u>f is subcontinuous, and</u>

(ii) <u>$C_K(f;x)$ is connected for each connected set K, and for each x in K.</u>

<u>Then f is in class C^{-3}; i.e., $\overline{f(K)}$ is connected for each connected subset K of X.</u>

Proof: Let K be a connected subset of X and suppose that $\overline{f(K)} = A \cup B$, where A and B are not empty and $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. Since this is a separation, there exist two open sets O_A and O_B containing A and B, respectively, such that $\overline{f(K)} \cap O_A \cap O_B = \emptyset$. Now, O_A and O_B must each intersect $f(K)$ also, since $f(K) \cap O_A = \emptyset$ implies $\overline{f(K)} \cap O_A = \emptyset$ and this is

a contradiction. Similarly, $f(K) \cap O_B \neq \emptyset$. Consequently, A and B must each have a non-empty intersection with f(K), for if $f(K) \cap A = \emptyset$, then $f(K) \cap O_A = \emptyset$. Similarly, $f(K) \cap B \neq \emptyset$. Let $A_1 = K \cap f^{-1}(A)$ and $B_1 = K \cap f^{-1}(B)$. Then $K = A_1 \cup B_1$ and $A_1 \cap B_1 = \emptyset$. But K is connected so we may, without loss of generality pick an x in $\overline{A_1} \cap B_1$. There is a net (x_d) in A_1 which converges to x and by (i) some subnet $(f(x_b))$ of $(f(x_d))$ converges to some point y. Since $f(x_b)$ is in A for each b, y is in $\overline{A}$ and since (x_b) converges to x, y is in $C_K(f;x)$. Finally, since $C_K(f;x) \cap A \neq \emptyset \neq C_K(f;x) \cap B$ and, by Theorem 8, $C_K(f;x) \subset A \cup B = \overline{f(K)}$, it follows that A and B separate $C_K(f;x)$. This contradicts (ii) and we conclude that $\overline{f(K)}$ is connected.

2-37 COROLLARY. <u>Let Y be a compact space. If $f: X \longrightarrow Y$ is a function such that $C_K(f;x)$ is connected for each connected set K and for each x in K, then f is in C^{-3}.</u>

2-38 REMARKS. In Bruckner, Ceder and Weiss [3], for an extended, real valued function f on R, $C^+(f;x)$ is defined to be the set of all extended reals y for which there exists a sequence (x_n) converging to x from the right such that $(f(x_n))$ converges to y. $C^-(f;x)$ is similarly defined. In Theorem 31 of [3] it is shown that f is in C^{-3} if and only if $C^+(f;x)$ and $C^-(f;x)$ are closed, connected sets for each x in X.

Note that in our notation we may write $C^+(f;x) =$

$C_{[x,\infty)}(f;x)$ and $C^-(f;x) = C_{(-\infty,\ x]}(f;x)$, where f is defined on the reals to the compact, Hausdorff space of extended reals. Thus f is in C^{-3} if and only if $C_A(f;x)$ is connected for every interval A which contains x as an end point. If the range space were not compact, this result would not be true. For example, the function $f: R \longrightarrow R$ (non-extended reals) defined by $f(x) = 1/x$, for $x \neq 0$ and $f(x) = 0$, for $x = 0$ is not in class C^{-3} but $C_A(f;x) = f(x)$ for each non-degenerate interval which contains x as an end point.

It can be shown that $C_{[x,\infty)}(f;x)$ and $C_{(-\infty,\ x]}(f;x)$ are connected for every x in X if and only if $C_A(f;x)$ is connected for every connected subset A of R and for every x in A. (See Theorem 14.) Theorem 36 is thus an extension of part of Theorem 3.1 in [3].

The conditions of Theorem 36 do not imply that f is connected. This is evident from the characterization of C^{-3} given in [3]. <u>In the following it is shown that hypothesis (i) is not essential for the conclusion of Theorem 36</u>.

2-39 EXAMPLE. Consider the function $f: R \longrightarrow R$ as defined in Example 1-18, which was shown to be in C^{-3}. The range space is not compact. Since the image of any bounded, non-degenerate interval I is dense in R it is possible to choose a sequence (x_n) in I which converges to some point x in I such that no subsequence of $(f(x_n))$ converges. Thus f is not subcontinuous. To see that condition (ii) is satisfied

consider $C_K(f;x)$, where K is a connected set and x is in K. By Theorem 4, $C_K(f;x) = \bigcap \{\overline{f(N \cap K)}: N \text{ is in } \mathcal{N}\}$, where $\mathcal{N}$ is a system of neighborhoods of x. If each N is an open interval, then each $N \cap K$ is an interval and $\overline{f(N \cap K)} = R$ since the image of $N \cap K$ is dense in R. Thus, the required partial cluster sets are connected.

QUESTION. In Theorem 36, is there a weaker condition than (i) which will suffice?

CHAPTER THREE

SOME APPLICATIONS TO TOPOLOGICAL VECTOR SPACES

All topological vector spaces considered here, denoted by E or F, will be Hausdorff (separated) and "topological vector space" will be abbreviated to "t.v.s.". Each t.v.s. E will have as its scalar field either the reals or the complex numbers with the usual topologies and if the scalar field is not specified, then either field applies. A linear operator on E to its scalar field is called a <u>linear functional</u>.

A subset A of E is called <u>balanced</u> if $tA \subset A$ for all scalars t such that $|t| \leq 1$. A subset A of E is called <u>absorbing</u> if for each vector x in E there exists an $\epsilon > 0$ such that tx is in A whenever t is a scalar such that $|t| \leq \epsilon$. The <u>line segment joining x and y</u> is the set $\{tx + (1-t)y: 0 \leq t \leq 1\}$ and is denoted by $[x, y]$. A set is <u>convex</u> if and only if it contains the line segment joining any two of its points.

If A is a balanced set, then it contains the zero element "0" and for any x in A the segment $[0, x] = \{tx: 0 \leq t \leq 1\}$ is in A. Thus $A = \bigcup\{[0, x]: x \text{ is in } A\}$. Each segment $[0, x]$ is arcwise connected, thus connected and since 0 is in $[0, x]$ for each x we conclude that A is a connected subset of E. The closure of a balanced set is balanced and if $T: E \longrightarrow F$ is a linear operator, then both T and T^{-1} take balanced sets to balanced sets.

In a t.v.s. E there exists a system $\mathcal{N}$ of neighborhoods of 0 such that

(i) every V in $\mathcal{N}$ is absorbing,

(ii) every V in $\mathcal{N}$ is balanced, and

(iii) for every V in $\mathcal{N}$ there exists a U in $\mathcal{N}$ such that $U + U \subset V$.

In the sequel, any neighborhood system $\mathcal{N}$ of 0 will be understood to have the above three properties. Now, for any x in E the sets of the form V + x as V runs through $\mathcal{N}$ form a system $\mathcal{N}_x$ of neighborhoods for the point x. Since the operation of translation is a homeomorphism each V + x is also connected. Thus any t.v.s. E is a locally connected topological space.

3-1 THEOREM. <u>Any linear operator $T: E \longrightarrow F$ is a weakly connected function</u>.

Proof: Let $\mathcal{N}$ denote the neighborhood system of 0. For each V in $\mathcal{N}$ and each x in E, $T(V) + T(x) = T(V + x)$ and is connected. Since all sets of the form V + x for V in $\mathcal{N}$ form a local base at x for each x, it follows that T is a locally connected function. By Theorem 1-10(b), T is a weakly connected function.

3-2 THEOREM. <u>A linear operator $T: E \longrightarrow F$ is a connected($\mathcal{B}$) function, where $\mathcal{B} = \{V + x: V \text{ is in } \mathcal{N}, x \text{ is in } E\}$</u>.

Proof: For each V in $\mathcal{N}$, $\overline{V}$ is balanced, and thus $T(\overline{V})$ is balanced and connected. Then $T(\overline{V + x}) = T(\overline{V} + x) = T(\overline{V}) + T(x)$

which is connected since the translation of a connected set is connected.

3-3 THEOREM. <u>A linear operator $T: E \longrightarrow F$ is continuous if and only if $T^{-1}(\text{bdry } U)$ is closed in E for each U of some neighborhood system $\mathcal{M}$ of 0 in F.</u>

Proof: If T is continuous, the result is immediate. To prove the converse note that by Remarks 1-27 T is continuous at 0 in E. From this, continuity of T on all of E follows.

3-4 REMARKS. The following is a comparison of the above result with another continuity theorem for linear operators.

A subset A of a t.v.s. is called <u>bounded</u> if and only if for each neighborhood U of 0 there is a real number $t > 0$ such that $A \subset tU$. The following statement may be found in Kelley and Namioka, [12], P. 45. "A sufficient condition that a linear function be continuous is that the image of some neighborhood of 0 be bounded. This condition is also necessary if the range space is pseudo-normable." It follows immediately from this that if $T: E \longrightarrow F$ is a linear operator and if the image of some neighborhood of 0 in E is bounded, then $T^{-1}(\text{bdry } U)$ is closed in E for each U in $\mathcal{M}$ of some system $\mathcal{M}$ of neighborhoods of 0 in F. The converse is not true since there exist t.v.s.'s E which have no bounded neighborhoods of 0; see [12], P. 55, Problem M. In such a case the identity function $i: E \longrightarrow E$ is continuous and linear but no neighborhood of 0 has a bounded image. In summary,

Theorem 3 gives a necessary and sufficient condition for continuity while the above stated condition is only sufficient in general.

3-5 COROLLARY. <u>A linear functional f on E is continuous if and only if $(Ref)^{-1}(0)$ is closed in E, where 0 is the zero of the reals and Ref denotes the real part of f.</u>

Proof: First note that $(Ref)^{-1}(0)$ is closed in E if and only if $(Ref)^{-1}(t)$ is closed for each real t. If the scalar field is the reals, then the result follows immediately from Theorem 3. If the scalar field is the complex numbers, then it is known that f is continuous if and only if its real part, Re(f), is continuous. The imaginary part of f, Im(f), can be expressed in terms of the real part by Im(f)(x) = -Re(f(ix)); see A. Wilansky, [27], p. 42.

If f is continuous, it is immediate that $(Ref)^{-1}(0)$ is closed.

3-6 REMARKS. The above corollary, which is well known, is a special case of Theorem 26 in Chapter 1 and depends on the fact that linear operators are weakly connected functions. It is not a consequence of the existing continuity theorems in the literature for connected functions or their existing generalizations.

3-7 EXAMPLE. Let ℓ denote the set of all real sequences $a = (a_n)$ such that $\sum |a_n| < \infty$ and define a norm on ℓ by $\|a\| = \sup |a_n|$. If $f(a) = \sum a_n$, then <u>f is a linear, discontinuous, real valued function</u> on ℓ. It is discontinuous since given any positive integer n, let $a = (a_i)$, where $a_i = 1$ for $i \leq n$ and $a_i = 0$ for $i > n$. Then $\|a\| = 1$, $f(a) = n$ and consequently f is unbounded.

The following theorem shows that discontinuous linear functionals are not even in class C^{-4}.

3-8 THEOREM. <u>A linear functional f on E is continuous if and only if it belongs to class C^{-4}</u>.

Proof: If f is continuous then f is connected and thus belongs to class C^{-4}. If f is not continuous, then it is known that $f^{-1}(0)$ is a dense, linear subspace of E and because of linearity is a connected subset. If $C = f^{-1}(0)$, we see that $f(\overline{C}) \not\subset \overline{f(C)}$, since $f \neq 0$, and thus f is not in C^{-4}.

3-9 REMARK. By Theorem 1 of D.E. Sanderson [21]; namely that f is in C^{-4} if and only if components of $f^{-1}(M)$ are closed if M is closed, we have the following extension of Theorem 3.1 of Pervin and Levine [19]. <u>If $f: X \longrightarrow Y$ is a monotone (point inverses are connected) function in class C^{-4}, then point inverses are closed</u>. This is not true in general for

weakly connected or for connected($\mathcal{B}$) functions as is seen by the example of a non-continuous linear functional. Such a function is weakly connected and connected($\mathcal{B}$) for some $\mathcal{B}$ and point inverses are connected but not closed.

3-10 THEOREM. Let $T: E \longrightarrow F$ be a linear operator. If either

(i) T is onto and in class C^{-4}, or

(ii) T is in $\mathcal{S}$,

then the graph of T is closed.

Proof: (i) Since T is in C^{-4} the inverse image of a closed set has closed components and, in particular, if the inverse image of a closed set is connected, it is also closed. Let $\mathcal{M}$ be a system of closed, balanced neighborhoods of 0 in $F = T(E)$. For each M in $\mathcal{M}$, $T^{-1}(M)$ is connected and closed so $\bigcap \overline{T^{-1}(M)} = \bigcap T^{-1}(M) = T^{-1}(\bigcap M) = T^{-1}(0)$. For any y in F, $T^{-1}(M + y) = T^{-1}(M) + T^{-1}(y)$ and is the inverse image of a closed set. Since $T^{-1}(M) + T^{-1}(y)$ is the sum of two connected sets and thus is itself connected it is also closed. Thus $\bigcap \overline{T^{-1}(M + y)} = \bigcap T^{-1}(M + y) = T^{-1}(\bigcap (M + y)) = T^{-1}(y)$. The intersections were taken over all M in $\mathcal{M}$. By Theorem 2-10 the graph of T is closed in $E \times F$.

(ii) Since F is a locally connected, regular space, by Theorem 2-20, if T is in class $\mathcal{S}$, the graph of T is closed.

3-11 THEOREM. A nearly continuous (see page viii) linear operator in class C^{-4} is continuous.

Proof: Let $T:E \longrightarrow F$ be the linear operator and let $\mathcal{M}$ be a neighborhood system of 0 in F consisting of closed, balanced sets. Since T is nearly continuous, $\overline{T^{-1}(M)}$ is a neighborhood of 0 in E and equals $T^{-1}(M)$ by the same argument as in part (i) of the above theorem. Therefore T is continuous at 0 and thus continuous.

3-12 COROLLARY. Let E be a second category t.v.s. A linear operator $T:E \longrightarrow F$ which belongs to class C^{-4} is continuous.

Proof: From Kelley and Namioka, [12], P. 97, T is nearly continuous.

QUESTION. Is a connected linear operator continuous in general?

3-13 REMARKS. If $T:E \longrightarrow F$ is a linear operator, then $C(T;x) = T(x) + C(T;0)$ and $C(T;tx) = tC(T;x)$ for each x and for each scalar t. The proofs are straightforward and depend on continuity of vector addition and scalar multiplication.

Since $C(T;0) = \bigcap \{\overline{T(N)} : N \text{ is in } \mathcal{N}\}$, where $\mathcal{N}$ is the system of balanced neighborhoods of 0 in E, it follows that this set is balanced and hence connected.

In the following, a continuity theorem is given for seminorms.

3-14 DEFINITION. A finite, real valued function P on a t.v.s.

E is called a <u>seminorm</u> if for all x and y in E and every scalar t,

(i) $P(tx) = |t|P(x)$ and

(ii) $P(x + y) \leq P(x) + P(y)$.

From this it follows that $P(0) = 0$, $P(x) \geq 0$ and $|P(x) - P(y)| \leq P(x - y)$. This last inequality implies that P is continuous if and only if P is continuous at 0 in E. Denote $\{t \text{ in } R : t \geq 0\}$ by R^+.

With every seminorm P on E there is associated a seminorm topology on E with respect to which the seminorm is continuous and with respect to which E is a t.v.s. (not necessarily Hausdorff).

3-15 THEOREM. <u>Any seminorm $P: E \longrightarrow R^+$ is a weakly connected function.</u>

Proof: Let $\mathcal{N}$ denote the neighborhood system of 0 in E consisting of balanced neighborhoods. Each U in $\mathcal{N}$ is also balanced in the topology $\mathcal{J}$ of the seminorm P on E and thus each U is $\mathcal{J}$-connected. Since $P: E \longrightarrow R^+$ is continuous with respect to $\mathcal{J}$, P(U) is connected in R^+. Similarly, $P(x + U)$ is a connected set in R^+ for each x in E and U in $\mathcal{N}$. Thus P is a locally connected function and by Theorem 1-10(b), P is weakly connected.

3-16 THEOREM. <u>A seminorm $P: E \longrightarrow R^+$ is continuous if and only if there exists a $t > 0$ such that $\overline{P^{-1}(t)}$ does not contain the zero of E.</u>

Proof: If P is continuous, then $P^{-1}(t)$ is closed for each t in R^+ and if $t > 0$, then 0 is not in $P^{-1}(t)$.

To prove the converse, it is known that P is continuous if and only if it is bounded on some open subset of E. Under the conditions of the theorem, for some $t > 0$ in R^+ there exists a balanced neighborhood U of 0 in E such that $U \cap P^{-1}(t) = \emptyset$. Since $P(U)$ is connected, does not contain t and does contain 0 it must be that $P(U)$ is bounded above by t. Thus P is a continuous function.

QUESTION. Does a connected seminorm have a closed graph?

3-17 EXAMPLE. <u>A seminorm need not be a connected function.</u> For any linear functional f on E the function P defined by $P(x) = |f(x)|$ for x in E is a seminorm. If f is not continuous, then P is not connected, as the following argument shows. Let $C = P^{-1}(0) = f^{-1}(0)$. Then C is a non-closed, connected, dense set in E and $P(\bar{C}) \not\subset \overline{P(C)}$. Thus P is not in C^{-4} and hence is not connected.

3-18 THEOREM. <u>A seminorm P in class C^{-4}, which is nearly continuous at 0 in E, is continuous.</u>

Proof: It is sufficient to show continuity of P at 0 in E. For any neighborhood $[0, t)$ of 0 in R^+ consider the neighborhood $[0, 1/2t)$. By near continuity $\overline{P^{-1}([0, 1/2t))}$ is a neighborhood of 0 and is furthermore connected since $P^{-1}([0, 1/2t))$ is convex. Since P is in C^{-4}, $P(\overline{P^{-1}([0, 1/2t))}) \subset \overline{PP^{-1}([0, 1/2t))}$

$\subset [0, 1/2t] \subset [0, t)$. Thus P is continuous at 0 in E.

3-19 REMARKS. In M.R. Mehdi [17], it is shown that if E is a second category t.v.s. and $P: E \longrightarrow R^+$ is a seminorm such that $\{x : P(x) \leq 1\}$ is a Baire set (S is a Baire set if $S = (G \setminus P) \cup R$, where G is open and both P and R are first category sets), then P is continuous. From this it follows that a seminorm in class C^{-4} on a second category t.v.s. is continuous. This is so since $\{x : P(x) \leq 1\} = P^{-1}([0, 1])$ is a convex, and thus connected, inverse image of a closed set, and by the property of class C^{-4}, is closed. A closed set is a Baire set and thus, P is continuous.

By the same arguments as above it is seen that a connected seminorm is lower semi-continuous; i.e., $P^{-1}((t, \infty))$ is open for each t in R^+. This is so since $P^{-1}((t, \infty)) = E - P^{-1}([0, t])$ and $P^{-1}([0, t])$ is closed.

From the known result that P is continuous if and only if it is bounded on some open set, it is possible to describe the cluster set of P at a point. If P is not continuous, then for any balanced neighborhood N of 0 in E, P(N) is unbounded, connected and contains 0. Thus, $P(N) = R^+$ for each N in $\mathcal{N}$ and $C(P;0) = R^+$. Similarly, $C(P;x)$ contains $[P(x), \infty)$ for each x in E.

3-20 DEFINITION. A real valued function f on a t.v.s. E is called convex if

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2},$$

for every x and y in E.

3-21 THEOREM. <u>If f is convex and U is a balanced neighborhood of 0, then $\overline{f(U)}$ is connected.</u>

Proof: We make use of the result of Theorem 4.5 of [3], that a convex function defined on the real line is in C^{-3} and thus in C^{-4}. First note that the restriction of f to any line $L = \{tx + (1 - t)y: t \text{ is in } R\}$ is a convex function. Now, $U = \bigcup\{[0, x]: x \text{ is in } U\}$ and $\overline{f(U)} = \overline{f(\bigcup [0, x])} \supset \bigcup \overline{f([0, x])}$, where the unions are over all x in U. By the result of [3] each $\overline{f([0, x])}$ is connected and since each such set also contains f(0), the union $\bigcup\{\overline{f([0, x])}: x \text{ is in } U\}$ is connected in R. Now if t is in $\overline{\bigcup\{f([0, x]): x \text{ is in } U\}}$, then t is a limit point of $\bigcup\{f([0, x]): x \text{ is in } U\}$ and thus a limit point of $\bigcup\{\overline{f([0, x])}: x \text{ is in } U\}$ which is an interval. Thus t is in $\overline{\bigcup\{\overline{f([0, x])}: x \text{ is in } U\}}$ and consequently $\overline{\bigcup\{f([0, x]): x \text{ is in } U\}} = \overline{\bigcup\{\overline{f([0, x])}: x \text{ is in } U\}}$. This means that $\overline{f(U)}$ is connected.

3-22 REMARKS. (a) By methods similar to those above it can be shown that $\overline{f(U + x)}$ is connected for each x in E and each U in $\mathcal{N}$.

(b) Every seminorm as well as every additive functional ($f(x + y) = f(x) + f(y)$) is a convex function. In M.R. Mehdi [18], it is shown that a convex function which is

bounded above on a non-empty open subset is continuous. The following theorem results from this.

3-23 THEOREM. <u>A convex function $f: E \longrightarrow R$ is continuous if and only if there is an interval I with $f(0) < \inf\{t: t \text{ is in } I\}$ and 0 is not in $\overline{f^{-1}(I)}$.</u>

Proof: If f is continuous, then $f^{-1}([a, b])$ is closed and does not contain 0 if $f(0) < a$. If 0 is not in $\overline{f^{-1}(I)}$, there exists a connected neighborhood U of 0 such that $U \cap \overline{f^{-1}(I)} = \emptyset$ and $\overline{f(U)}$ is connected. Now, "$f(0)$ is in $\overline{f(U)}$", "$f(0) < \inf\{t: t \text{ is in } I\}$" and "$\overline{f(U)} \cap I = \emptyset$" together imply that $\overline{f(U)}$ is bounded above by each element of I. Thus f(U) is bounded above and f is consequently continuous.

3-24 REMARKS. If a convex function $f: E \longrightarrow R$ is not continuous at 0 in E, then, for each N in $\mathcal{N}$, f(N) is not bounded above. Thus C(f;0), which is $\bigcap\{\overline{f(N)}: N \text{ is in } \mathcal{N}\}$ contains $[f(0), \infty)$. Similarly, C(f;x) is unbounded for each x in E. From this it follows that if the graph of f is closed, f must be continuous. See the Question posed before Example 17.

In F.B. Jones [11], Theorem 1, it is stated that for the case when f is an additive function on the real line if f is discontinuous, then the graph of f is dense in the plane. Theorem 5 of [11] indicates that a connected, convex function need not be continuous.

CHAPTER FOUR

NONCONTINUOUS MULTIFUNCTIONS

Many of the theorems from Chapters one and two can be extended to multifunctions. The proof of each theorem stated here for multifunctions is identical to the proof of the corresponding theorem for functions and hence is not given.

A multifunction F from a topological space X to a topological space Y is a set valued relation taking each element x in X to a nonempty subset F(x) of Y. For any subset A of X, $F(A) = \bigcup\{F(x): x \text{ is in } A\}$ and, for any subset B of Y, $F^{-1}(B) = \{x \text{ in } X: F(x) \cap B \neq \emptyset\}$. F is upper semi-continuous (u.s.c.) at x in X if and only if for each open subset V of Y such that $F(x) \subset V$ there is a neighborhood N of x such that $F(N) \subset V$. If F is u.s.c. at each x in X, then F is said to be u.s.c. The graph of a multifunction $F: X \longrightarrow Y$ is $\{(x,y) \text{ in } X \times Y: y \text{ is in } F(x)\}$ and is denoted by G(F). The multifunction $F_G: X \longrightarrow X \times Y$ defined by $F_G(x) = \{x\} \times F(x)$ is called the graph multifunction of F. A multifunction F is called point connected (compact) if F(x) is connected (compact) for each point x in X.

Just as for functions a multifunction F is called weakly connected if F(K) is connected for every connected, open set K. F is called connected if F(K) is connected for each connected set K and is called a connectivity multifunction

if F_G is connected. The class of connectivity multifunctions is also denoted by C^{-1} and the class of connected multifunctions by C^{-2}. A multifunction F is said to be in class C^{-3} if and only if $\overline{F(K)}$ is connected for every connected set K and F is in class C^{-4} if and only if $F(\overline{K}) \subset \overline{F(K)}$ for every connected set K. In the case of functions, for n = 1, 2, 3, 4, the inclusions $C^{-(n-1)} \subset C^{-n}$ hold and even though the inclusions do not all hold in general for multifunctions it is still convenient to use the C^{-n} notation. Under certain conditions some inclusions do hold for multifunctions. In [R.T. Douglass, Connectivity multifunctions and the pseudo-quotient topology, University of Kansas, Dissertation, 1967] it is shown that $C^{-1} \subset C^{-2}$ and if F is a point connected, point compact multifunction which is u.s.c., then F is in C^{-1}. The inclusion $C^{-2} \subset C^{-3}$ clearly holds, but the inclusion $C^{-3} \subset C^{-4}$ does not hold as the multifunction $F: I \longrightarrow I$, defined by $F(x) = 0$ for $0 \leq x < 1$ and $F(1) = I$, shows.

Let $\{S_d : d \text{ is in } D\}$ be a net of sets in X, where D is a directed set. If the directed set is understood, the net will be denoted by (S_d) or $(S(d))$. Superior and inferior limits are defined as follows.

$\overline{\lim}_d S_d = \{$x in X: for any open subset G of X about x and for any d in D there exists a $d' > d$ such that $G \cap S_{d'} \neq \emptyset\}$.

$\underline{\lim}_d S_d = \{$x in X: for any open set G about x there exists a d

in D such that $G \cap S_{d'} \neq \emptyset$ for all $d' > d\}$.

A net of sets is defined to be frequently (eventually) in a set S just as a net of points is frequently (eventually) in a set S.

Property H, as in Definition 1-22, and the notion of peripheral F-normality are extended verbatim to multifunctions.

The following theorem is an extension of Theorem 1-24 to multifunctions and becomes a generalization of Theorem B of G.T. Whyburn [25] in the sense that the multifunction F is weakly connected rather than connected and "Property H" is more general than "peripheral F-normality".

4-1 THEOREM. <u>If X is locally connected and $F: X \longrightarrow Y$ is a weakly connected function with Property H, then F is u.s.c.</u>

The extension of Theorem 1-26 to multifunctions is given below. This theorem generalizes Theorem 1 of R.E. Smithson [22]. Furthermore, if the range space Y is regular, and F is point compact, then this theorem becomes a corollary of the above theorem.

4-2 THEOREM. <u>If X is locally connected and $F: X \longrightarrow Y$ is a weakly connected function, then F is u.s.c. if and only if F^{-1}(bdry G) is closed for each open subset G of Y.</u>

Now, if Y, in the above theorem, is a regular space and F is point compact, then F has the property H. To see this, let K be a closed set in the complement of some point image F(x). By regularity, for each point y in F(x) there exists a closed neighborhood N_y of y contained in Y − K. By compactness of F(x) a finite number of such neighborhoods N_{y^i}, i = 1, 2, ···, n cover F(x). If U is the union of this finite cover, then bdry U separates F(x) and K. Furthermore, x is not contained in $\overline{F^{-1}(\text{bdry } U)}$.

4-3 REMARKS. The following is also true and the proof is similar to that of Theorem 2.

<u>If X is locally connected and if $F: X \longrightarrow Y$ is a weakly connected multifunction such that for each open set G about F(x), $\overline{F^{-1}(\text{bdry } G)}$ does not contain x, then F is u.s.c. at x.</u>

The extension of Theorem 2-27 to multifunctions is as follows.

4-4 THEOREM. <u>Let X be locally connected and Y rim-compact. If $F: X \longrightarrow Y$ is a weakly connected multifunction which is point compact and has a closed graph, then F is u.s.c.</u>

Cluster sets and some of the related theorems in Chapter 2 are now extended to multifunctions.

4-5 DEFINITION. For any multifunction $F: X \longrightarrow Y$ and for any

x in X let $C(F;x)$ be the set of all y in Y for which there exists a net (x_d) converging to x such that y is in $\overline{\lim}_d F(x_d)$. This set is called the <u>cluster set of F at x</u> and if no confusion arises it is called a <u>cluster set</u>.

If A is any subset of X, let $C_A(F;x)$ denote the set of all y in Y for which there exists a net (x_d) in A converging to x such that y is in $\overline{\lim}_d F(x_d)$. This set is called the <u>partial cluster set of F at x with respect to A</u>.

4-6 REMARKS. The following results follow readily from the definitions.

(a) <u>Let X be a space in which all components are open. If partial cluster sets with respect to connected sets are connected, then the cluster sets are connected.</u>

(b) <u>For any subset K of X, $C_K(F;x)$ is a subset of $\overline{F(K)}$ for each x in X.</u>

(c) <u>By a proof analogous to that of Theorem 2-4 it follows that $C_K(F;x) = \bigcap\{\overline{F(N \cap K)} : N \text{ is in } \mathcal{N}\}$, for any subset K of X and any x in X, where $\mathcal{N}$ represents a neighborhood system of x.</u>

4-7 DEFINITION. For any multifunction $F:X \longrightarrow Y$ and for any y in Y let $T(F;y)$ be the set of all x in X for which there exists a net (x_d) converging to x such that y is in $\overline{\lim}_d F(x_d)$.

By a proof analogous to the proof of Theorem 2-6 it can be shown that $T(F;y) = \bigcap\{\overline{F^{-1}(N)} : N \text{ is in } \mathcal{N}_y\}$, where $\mathcal{N}_y$ is a neighborhood system of y.

The following are extensions of Theorems 2-9 and 2-10.

4-8 THEOREM. For any multifunction $F: X \longrightarrow Y$, $G(F)$ is closed if and only if $C(F;x) = F(x)$ for every x in X.

4-9 THEOREM. For any multifunction $F: X \longrightarrow Y$, $G(F)$ is closed if and only if $T(F;y) = F^{-1}(y)$ for every y in Y.

4-10 DEFINITION. A multifunction $F: X \longrightarrow Y$ is said to be subcontinuous if whenever a net (x_d) converges to some x in X, then for any net (y_d), with y_d in $F(x_d)$, for each d, there is a subnet which converges to some y in Y.

The following theorem is an extension to multifunctions of Theorem 2-16.

4-11 THEOREM. Let X be locally connected and let Y be a normal space. If $F: X \longrightarrow Y$ is a multifunction such that

(i) F is weakly connected (or in C^{-3}) and

(ii) F is subcontinuous,

then $C(F;x)$ is connected for each x in X.

4-12 COROLLARY. Let X be locally connected and let Y be a compact, Hausdorff space. If $F: X \longrightarrow Y$ is a weakly connected multifunction (or is in C^{-3}), then $C(F;x)$ is connected for each x in X.

The extension of Theorem 2-30 to multifunctions is as follows.

4-13 THEOREM. Let X be locally connected and let Y be compact Hausdorff. A weakly connected, point closed multifunction $F: X \longrightarrow Y$ is u.s.c. if and only if $C(F;x)$ is the union of a countable number of disjoint, closed sets one of which is $F(x)$.

The following is the extension of Theorem 2-20.

4-14 THEOREM. Let Y be a locally connected, regular space and let $F: X \longrightarrow Y$ be a multifunction satisfying the property that $F^{-1}(\bar{K}) \supset \overline{F^{-1}(K)}$ for every connected subset K of Y. Then the graph of F is closed in $X \times Y$.

Theorems 2-31 and 2-36 extend to multifunctions in the following manner.

4-15 THEOREM. Suppose that a point connected multifunction $F: X \longrightarrow Y$ has the following properties.

(i) F is subcontinuous.

(ii) $C(F;x)$ is connected for each x in X.

(iii) For each nondegenerate, connected subset C of X and for each x in C, $C(F;x) \subset F(C)$.

Then F is a connected multifunction.

4-16 THEOREM. Let $F: X \longrightarrow Y$ be a point connected multifunction with the following properties.

(i) F is subcontinuous.

(ii) $C_K(F;x)$ is connected for each connected set K.

Then $\overline{F(K)}$ is connected for each connected set K; i.e., F is in C^{-3}.

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